WAVE PATTERNS IN SUPERSONIC FLOW OF BOSE-EINSTEIN CONDENSATE PAST AN OBSTACLE

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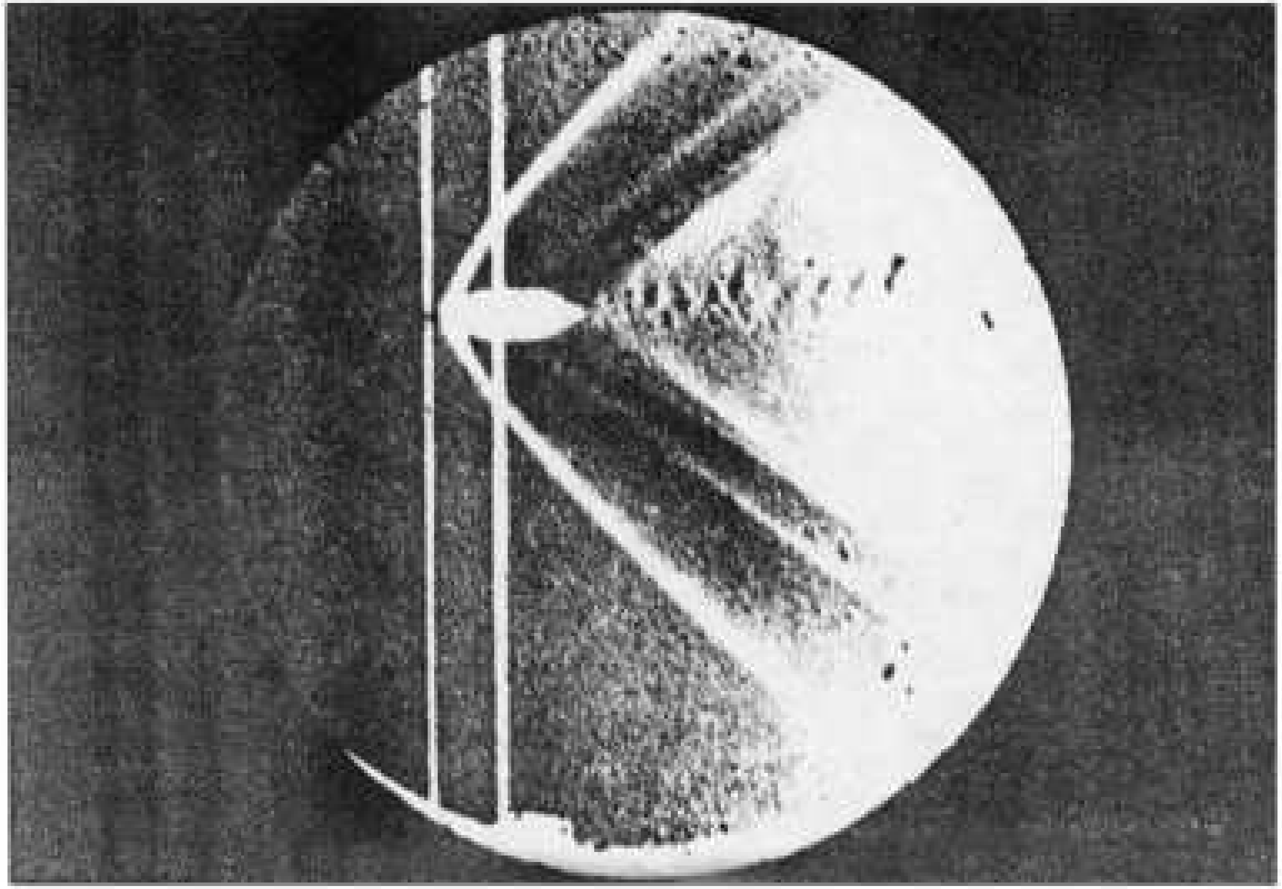
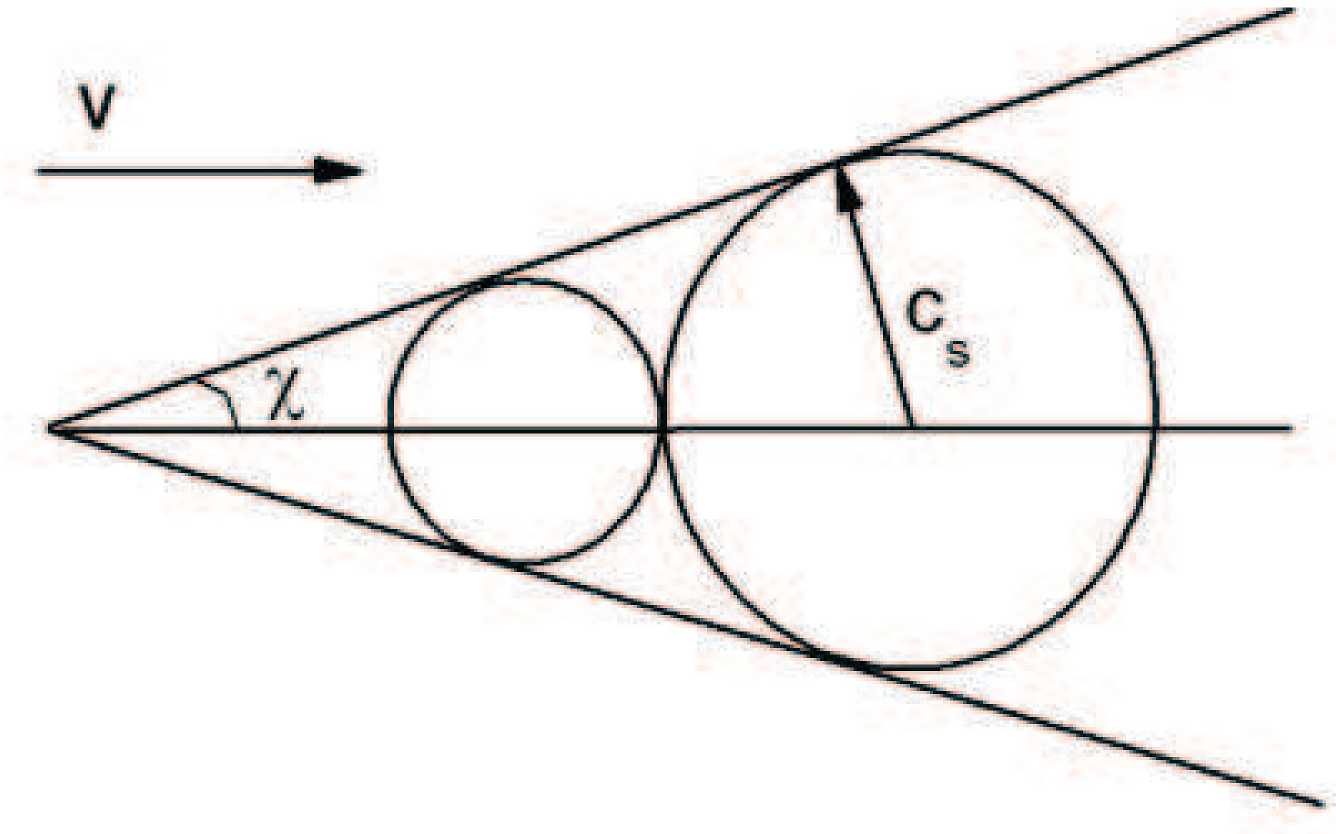
Outline

- **Shocks in dissipative media: Ernst Mach experiment on shocks in the air**
- **Shocks in dispersive media: Eric Cornell experiment on shocks in BEC**
- **Dispersion relation for linear waves in BEC and its consequences**
- **Kelvin's "ship waves" in BEC outside the Mach cone**
- **Oblique dark solitons in BEC inside the Mach cone**
- **Dispersive shocks**
- **Conclusions**

Shocks in dissipative media: Ernst Mach experiment on shocks generated by a bullet moving through the air

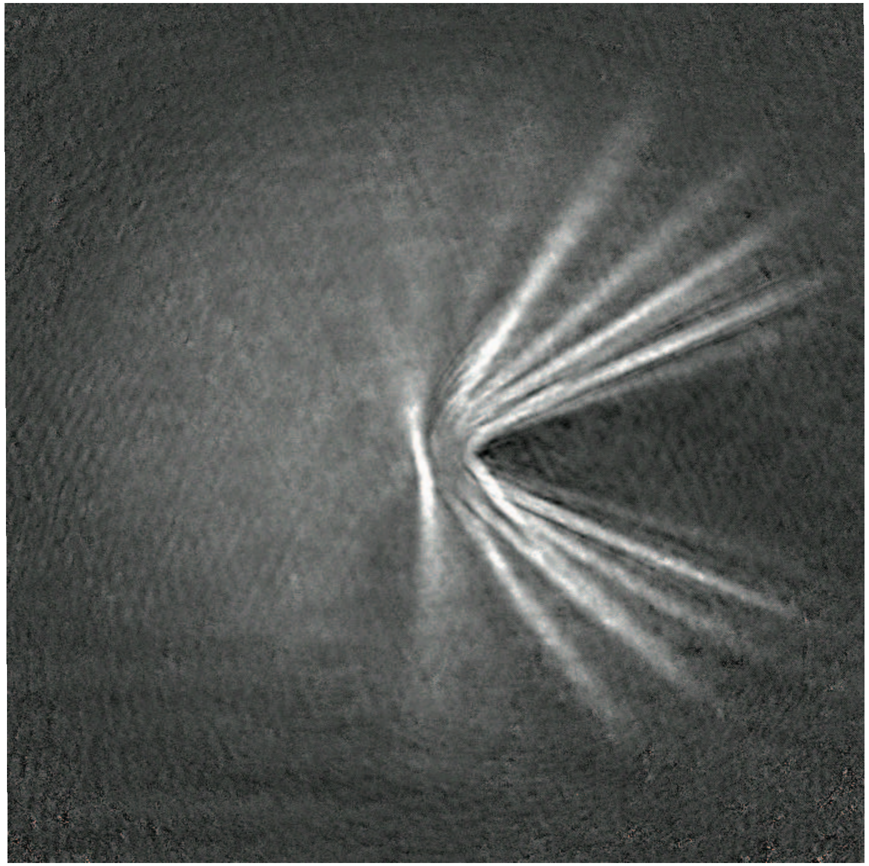


Formation of the Mach cone in medium without dispersion



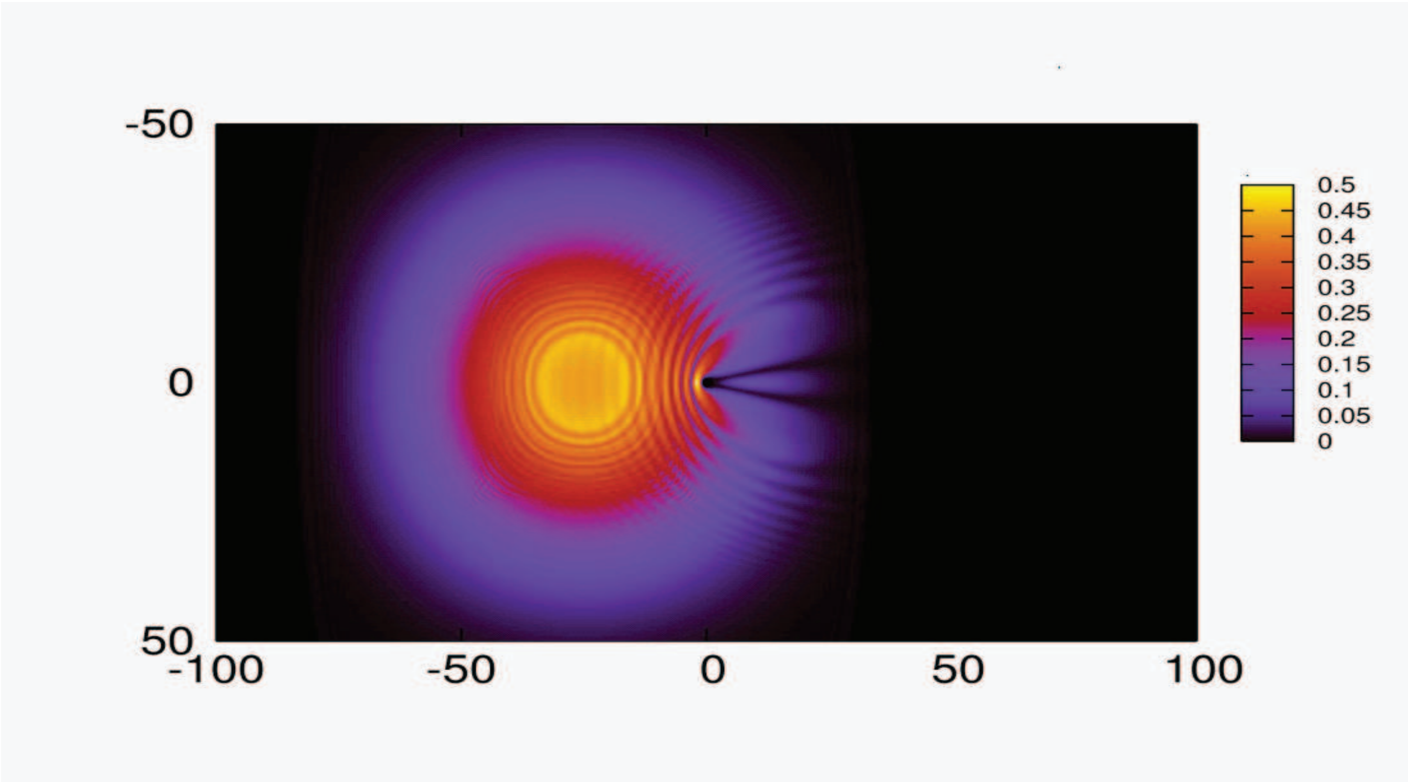
$\sin\chi = \frac{c_s}{v} = \frac{1}{M}$ and $M = \frac{v}{c_s}$ is the Mach number

Shocks in dispersive media: Eric Cornell experiment on shocks generated by flow of expanding BEC past an obstacle



(From: E.A. Cornell, in “Conference on Nonlinear Waves, Integrable Systems and Their Applications”, Colorado Springs, June 2005; http://jilawww.colorado.edu/bec/papers.html)

Numerical simulation

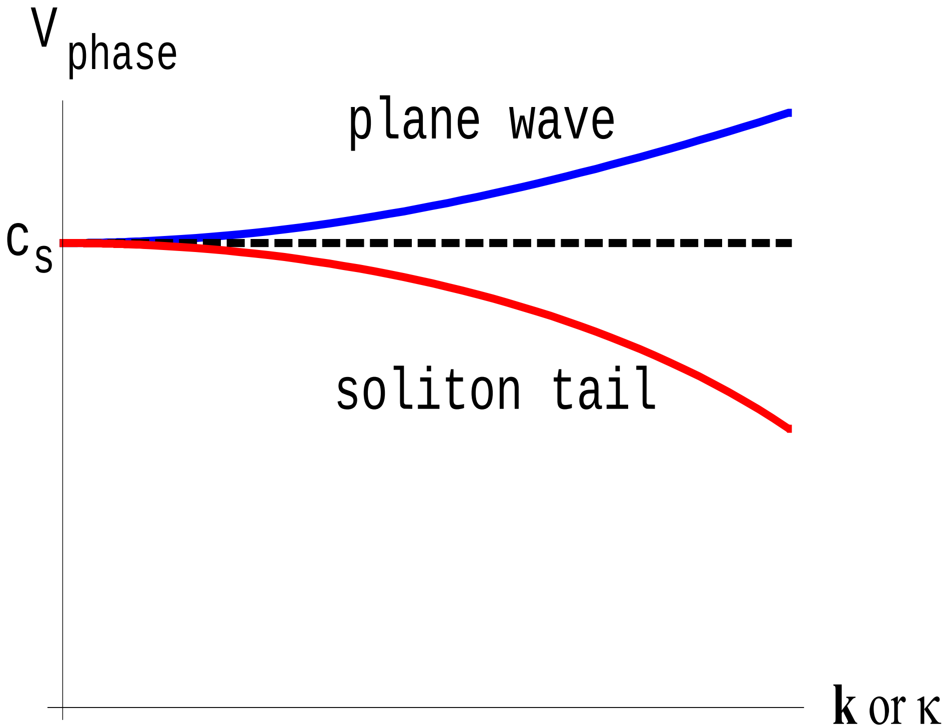


Evolution of BEC is described by the Gross-Pitaevskii equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + U(\mathbf{r})\psi + g|\psi|^2\psi$$

with the condensate located initially in the bump near the origin of the coordinate system.

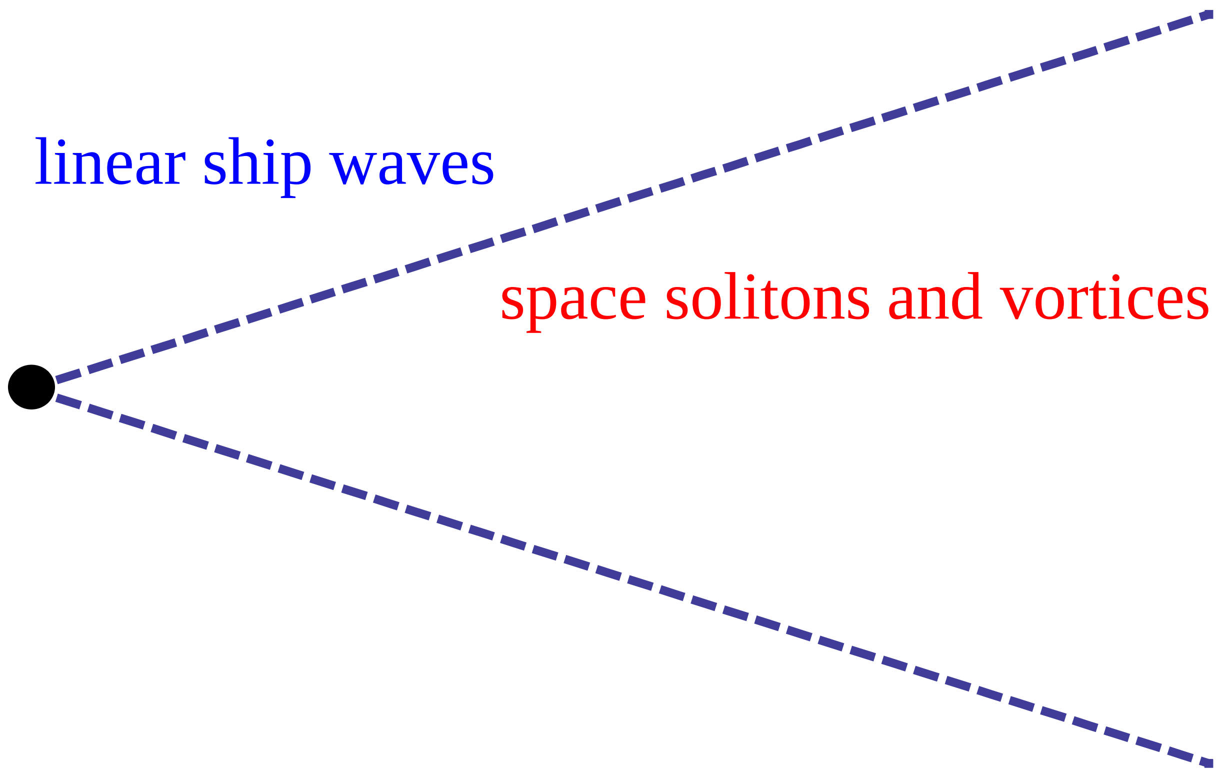
Phase velocities of linear waves and soliton's tails



Phase velocity $V_{phase} = \omega/k$ of linear plane waves and velocity of solitons's tails as functions of wave number k or inverse half-width of a soliton are given, respectively, by

$$V_{phase} = \frac{\omega}{k} = \sqrt{1 + \frac{k^2}{4}} \quad \text{or} \quad V_s = \sqrt{1 - \frac{\kappa^2}{4}}.$$

Regions with wave structures of different origin



Linear "ship waves" are located outside the Mach cone and dark space solitons and vortices are located inside it.

The theory of "ship waves" in BEC

(Yu.G. Gladush, G.A. El, A. Gammal, A.M. Kamchatnov, Phys. Rev. A **75,** 033619 (2007);
Yu.G. Gladush, A.M. Kamchatnov, arXiv: 0705.3180 [nlin.PS]; to be published in JETP)

We transform the Gross-Pitaevskii equation to hydrodynamic form by means of the substitution

$$\psi(\mathbf{r},t)=\sqrt{n(\mathbf{r},t)}\exp\left(\frac{i}{\hbar}\int^{\mathbf{r}}\mathbf{u}(\mathbf{r}',t)d\mathbf{r}'\right),$$

and cast it to non-dimensional variables. As a result we arrive at the system

$$n_t+(nu)_x+(nv)_y=0,$$

$$u_t+uu_x+vu_y+n_x+\left(\frac{n_x^2+n_y^2}{8n^2}-\frac{n_{xx}+n_{yy}}{4n}\right)_x=-V_xe^{\varepsilon t},$$

$$v_t+uv_x+vv_y+n_y+\left(\frac{n_x^2+n_y^2}{8n^2}-\frac{n_{xx}+n_{yy}}{4n}\right)_y=-V_ye^{\varepsilon t}.$$

Here $n(\mathbf{r},t)$ has a meaning of the condensate density and $\mathbf{u}(\mathbf{r},t)$ of its flow velocity.

It is assumed that the potential of the obstacle increases exponentially slowly with time:
$V(\mathbf{r},t)=V(\mathbf{r})\exp(\varepsilon t);\ \varepsilon\to 0.$

We are interested in linear waves propagating in a uniform flow with $n_0 = 1$, $u_0 = M$, $v_0 = 0$. Therefore we introduce new variables

$$n = 1 + n_1, \quad u = M + u_1, \quad v = v_1$$

and linearize the system with respect to small deviations $n_1,\ u_1,\ v_1$ from a uniform flow. As a result we arrive at a linear system

$$n_{1,t} + u_{1,x} + Mn_{1,x} + v_{1,y} = 0,$$

$$u_{1,t} + Mu_{1,x} + n_{1,x} - \tfrac{1}{4}(n_{1,xxx} + n_{1,xyy}) = -V_x e^{\varepsilon t},$$

$$v_{1,t} + Mv_{1,x} + n_{1,y} - \tfrac{1}{4}(n_{1,xxy} + n_{1,yyy}) = -V_y e^{\varepsilon t}.$$

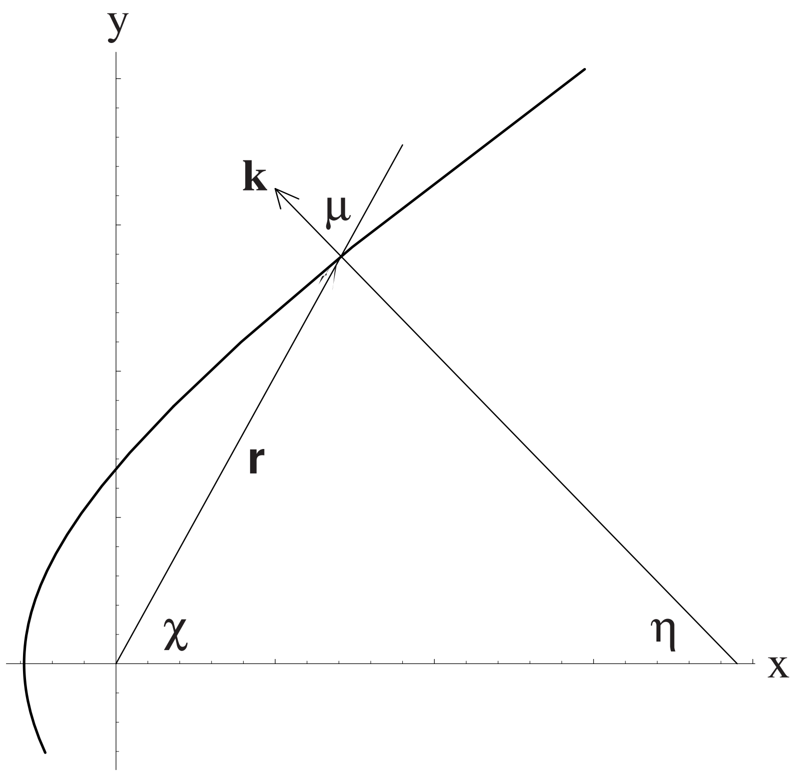
Fourier transform for spatial variables yields the solution

$$n_1 = -e^{\varepsilon t} \iint \frac{k^2 \Phi(k_x, k_y) e^{i(k_x x + k_y y)}}{(\varepsilon + ik_x M)^2 + k^2(1 + k^2/4)} \frac{dk_x dk_y}{(2\pi)^2},$$

where

$$\Phi(\mathbf{k}) = \int V(\mathbf{r}) e^{-i\mathbf{kr}} d\mathbf{r}.$$

We take $V(\mathbf{r}) = V_0\delta(\mathbf{r})$, hence $\Phi(\mathbf{k}) = V_0$, and introduce the polar coordinates



so that

$$x = r\cos\chi, \quad y = r\sin\chi; \qquad k_x = -k\cos\eta, \quad k_y = k\sin\eta.$$

Then simple transformation gives

$$n_1 = \frac{V_0}{\pi^2} e^{\varepsilon t} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{k e^{-ikr\cos(\chi+\eta)} dk d\eta}{k^2 - k_0^2 - i\delta \cos\eta},$$

where $\delta = 8M\varepsilon/k$ is a small positive quantity and $k_0 = 2\sqrt{M^2\cos^2\eta - 1}$. For large distance from the obstacle, $kr \gg 1$, this integral can be estimated with the use of the method of stationary phase which results in

$$n_1 = V_0 \sqrt{\frac{2k}{\pi r} \frac{[(M^2-2)k^2 + 4(M^2-1)]^{1/4}}{[(M^2-2)k^2 + 6(M^2-1)]^{1/2}}} \cos\left(kr\cos\mu - \frac{\pi}{4}\right),$$

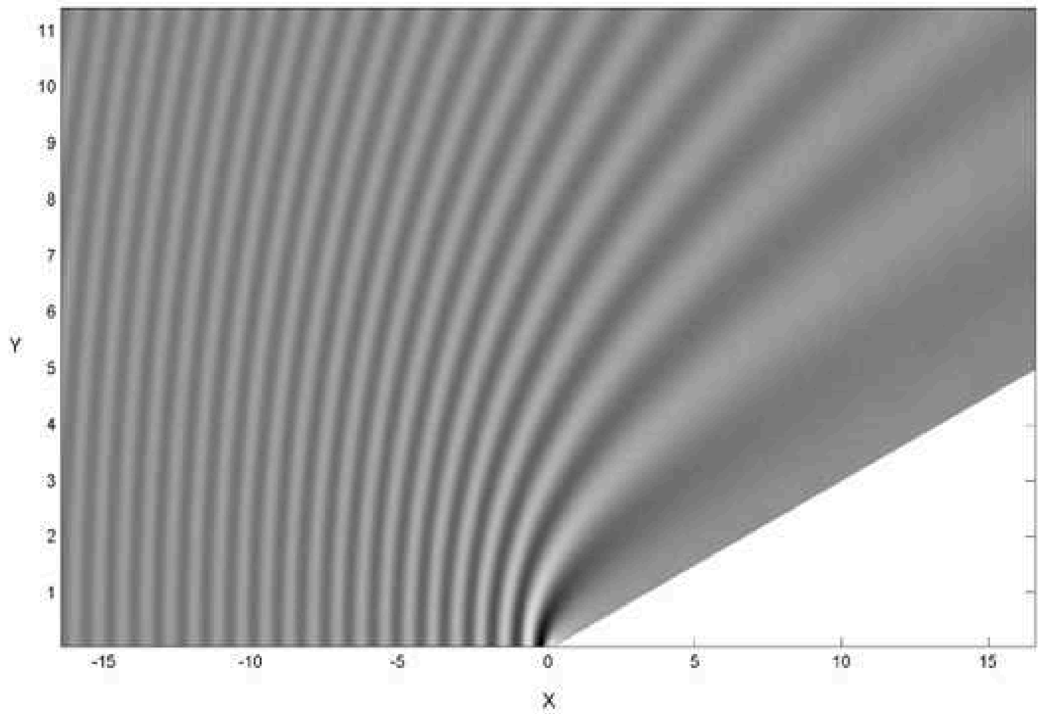
where

$$k = 2\sqrt{M^2\cos^2\eta - 1}$$

and

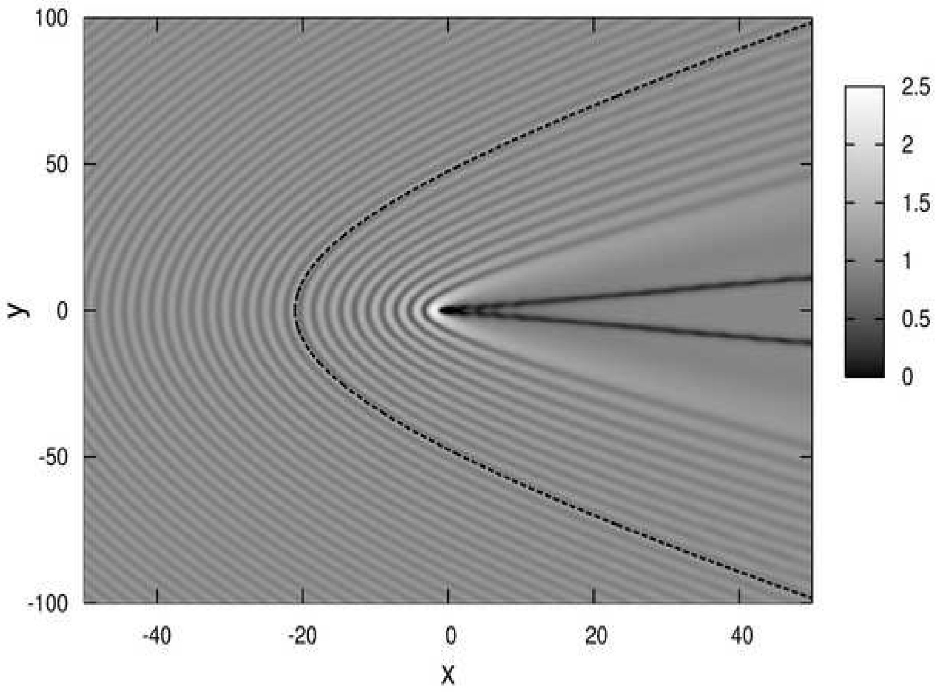
$$\tan\chi = \frac{(1+k^2/2)\tan\eta}{M^2 - (1+k^2/2)}, \quad \tan\mu = \frac{2M^2}{k^2}\sin 2\eta.$$

“Ship wave” structure



The wave structure outside the Mach cone; it is calculated analytically for the case of the Mach number $M = 4$.

Geometric form of wavecrests



Geometric form of curves of constant phase is defined by parametrical formulae

$$x = r\cos\chi = \frac{4\phi}{k^3}\cos\eta(1 - M^2\cos 2\eta), \quad y = r\sin\chi = \frac{4\phi}{k^3}\sin\eta(2M^2\cos^2\eta - 1),$$

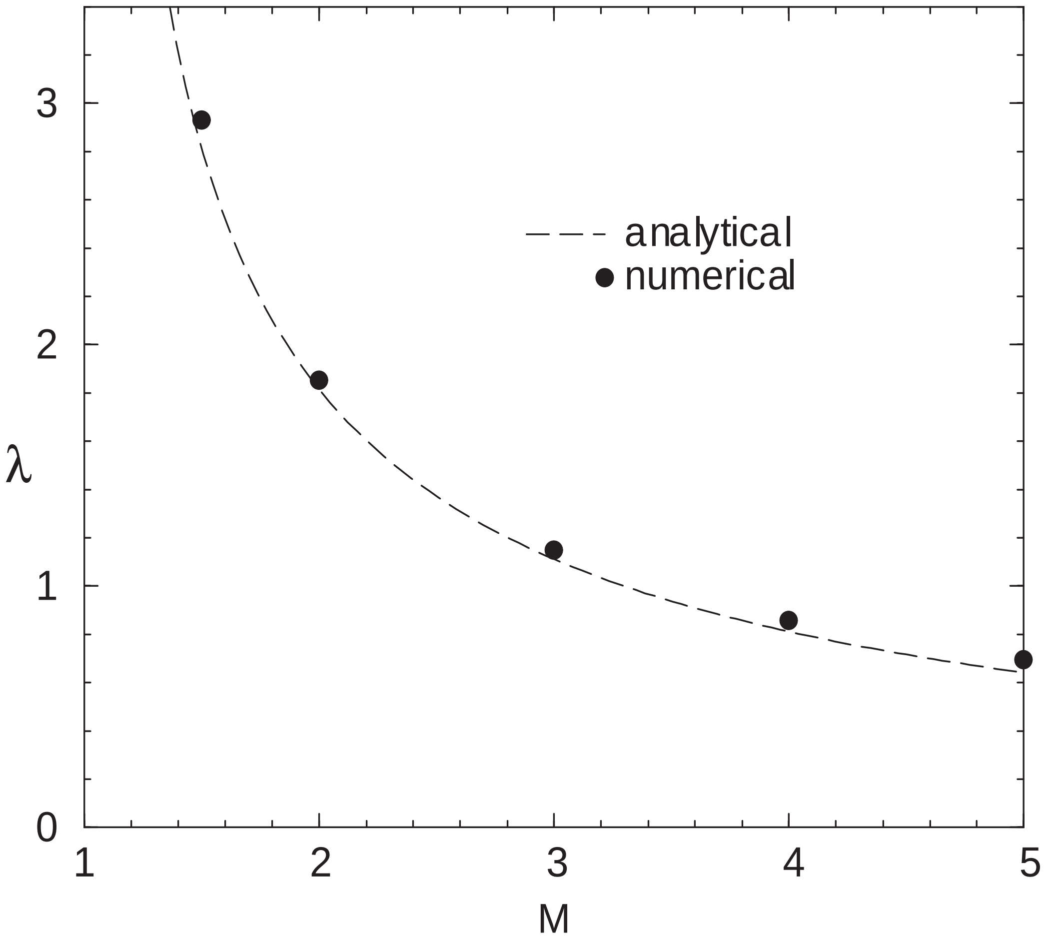
where k is a function of η and the parameter η varies in the interval

$$-\arccos(1/M) \leq \eta \leq \arccos(1/M).$$

Dependence of the wavelength on the Mach number

At $y = 0, \;\; x < 0$ the wavelength is equal to

$$\lambda = \frac{2\pi}{k} = \frac{\pi}{\sqrt{M^2 - 1}}.$$



Oblique spatial soliton

(G.A. El, A. Gammal, A.M. Kamchatnov, Phys. Rev. Lett. **97,** 180405 (2006))

Now we have to look for a stationary solution of a nonlinear Gross-Pitaevskii system

$$\begin{aligned}(nu)_x + (nv)_y &= 0,\\ uu_x + vu_y + n_x + \left(\frac{n_x^2+n_y^2}{8n^2} - \frac{n_{xx}+n_{yy}}{4n}\right)_x &= 0,\\ uv_x + vv_y + n_y + \left(\frac{n_x^2+n_y^2}{8n^2} - \frac{n_{xx}+n_{yy}}{4n}\right)_y &= 0,\end{aligned}$$

which is subject to the boundary conditions condition that the BEC flow is uniform at infinity:

$$n = 1, \quad u = M, \quad v = 0 \quad \text{at} \quad |x| \to \infty,$$

When the solution is looked for in the form $n = n(\theta)$, $u = u(\theta)$, $v = v(\theta)$, where $\theta = x - ay$, and a denotes a slope of the soliton with respect to y axis, the system reduces to the equation

$$\frac{1}{4}(1 + a^2)(n'^2 - 2nn'') + 2n^3 - (2 + p)n^2 + p = 0,$$

where

$$p = M^2/(1 + a^2).$$

It can be easily solved to give

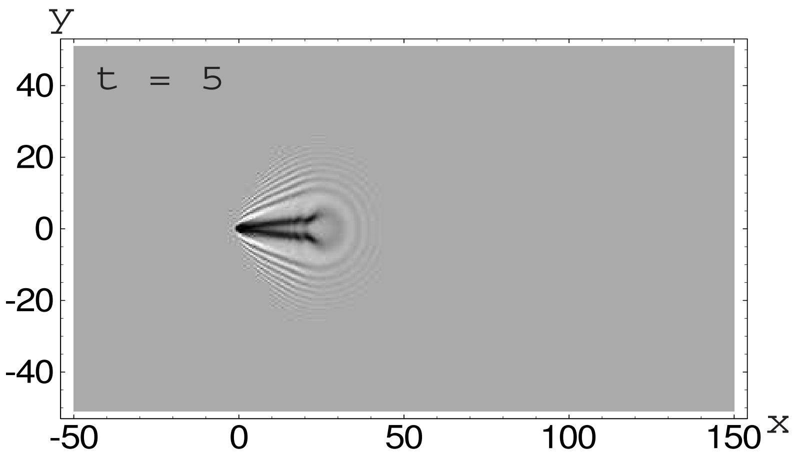
$$n(\theta) = 1 - \frac{1 - p}{\cosh^2[\sqrt{1 - p}\,\theta/\sqrt{1 + a^2}]}.$$

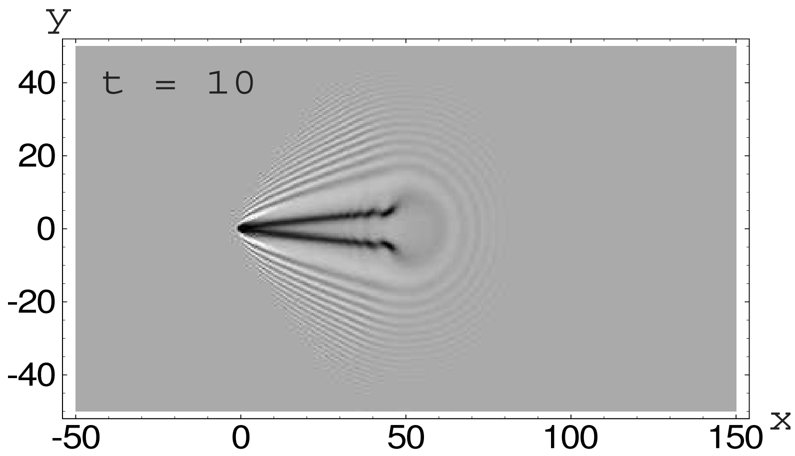
A small amplitude limit is achieved when $1 - p \ll 1$. It describes shallow dark KdV solitons located close to the Mach cone.

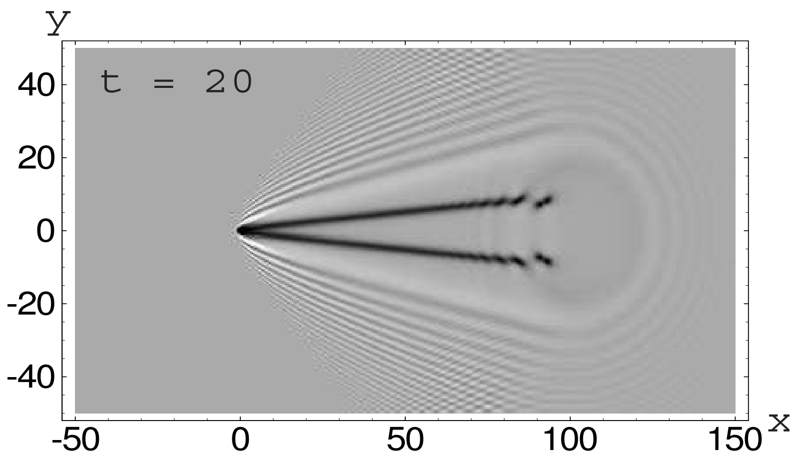
Large slopes $a^2 \gg 1$ correspond to deep dark NLS solitons located close to the flow axis.

Generation of oblique spatial soliton by an obstacle

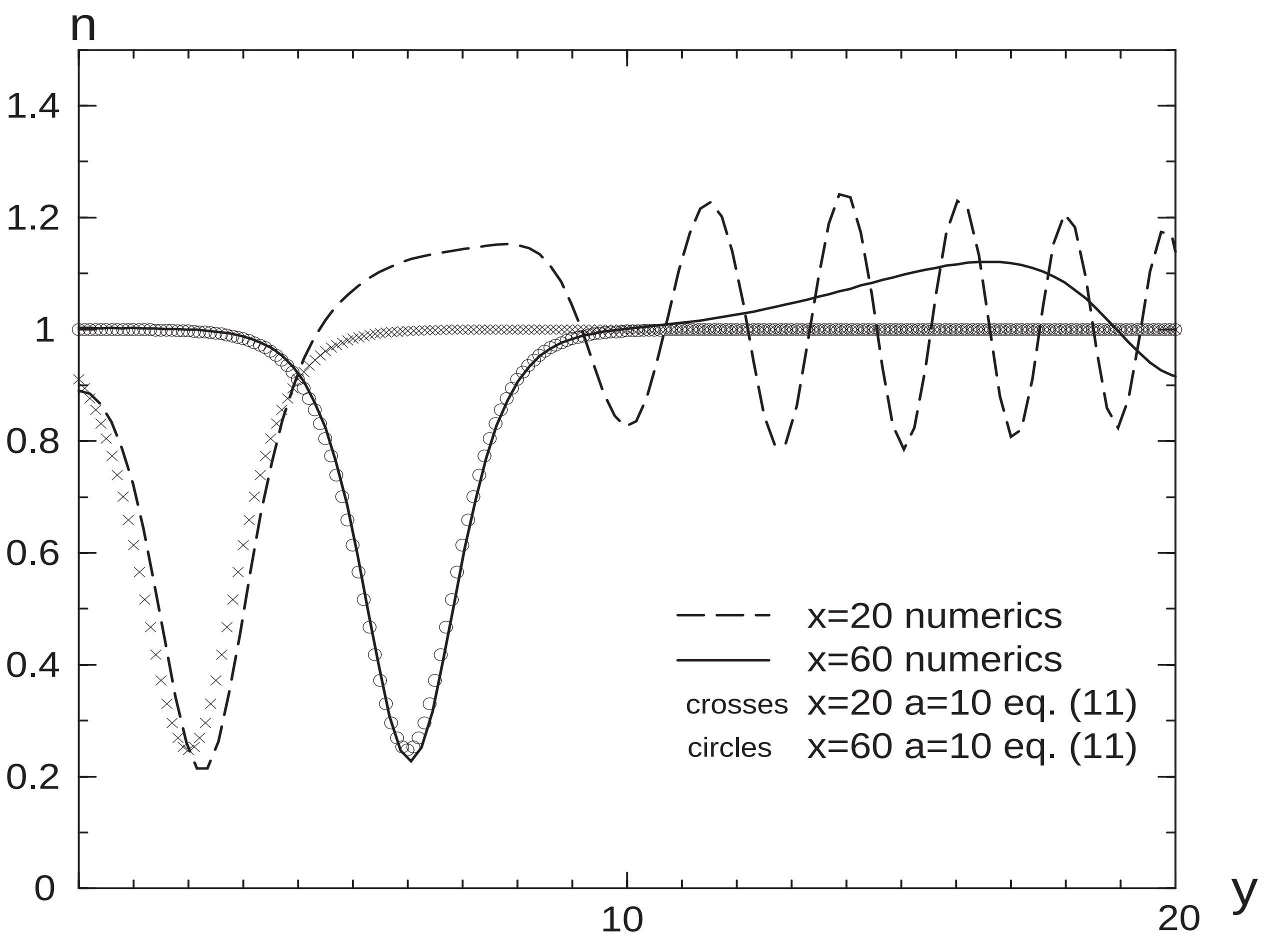
Numerical solution of time-dependent Gross-Pitaevskii equation yields the pictures:



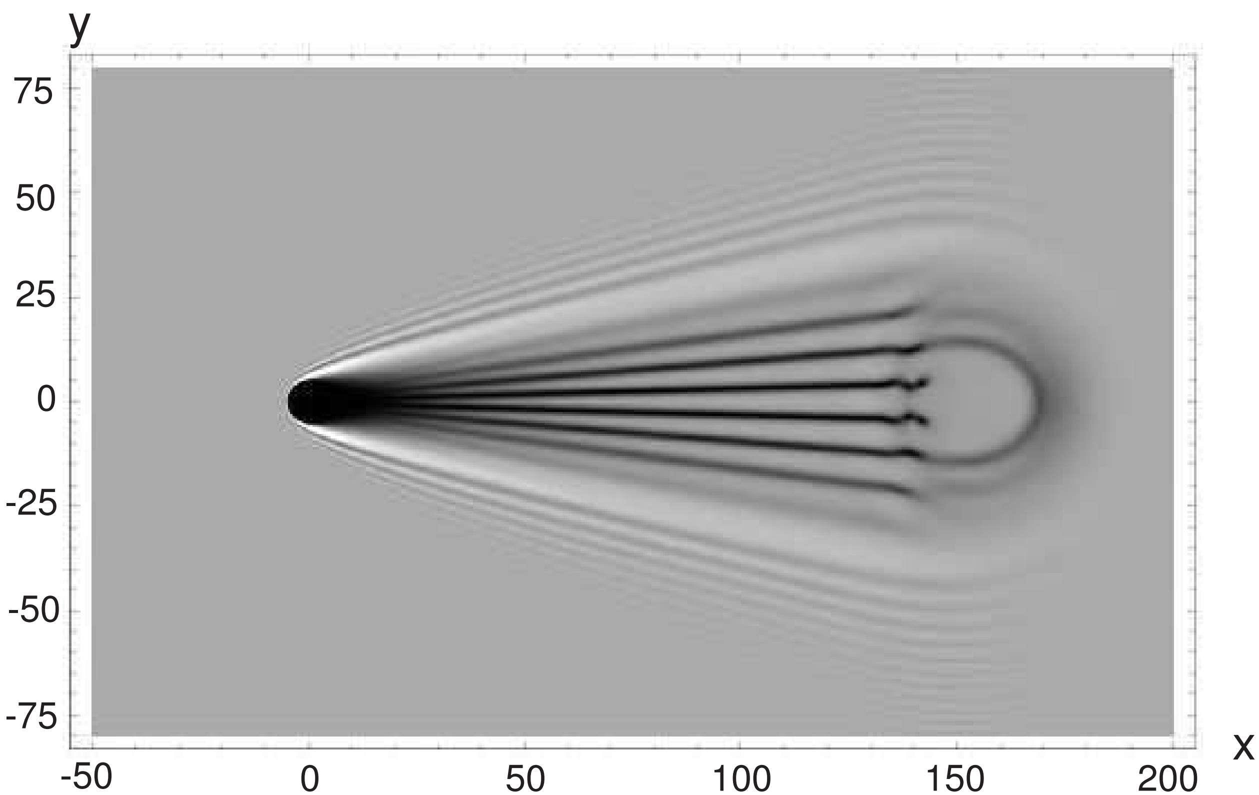




Comparison of numerical density profiles with analytical solution



Dispersive shocks



Conclusions

- Supersonic flow of BEC past an obstacle generates various wave structures
- These structures are located in different space regions separated by the Mach cone corresponding to the sound velocity of waves with infinite wavelength
- Outside the Mach cone the “ship waves” are generated. They can be described by a linear theory far enough from the obstacle.
- Inside the Mach cone vortices and spatial dark solitons are generated which are described by nonlinear solutions of the Gross-Pitaevskii equation.
- The theory explains qualitatively the experiments on generation of wave patterns in the flow past an obstacle of expanding BEC released from a trap.